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EXISTENCE RESULTS FOR VECTOR VARIATIONAL-LIKE INEQUALITIES

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Abstract

In this paper, we consider and study a class of vector variational-like inequalities in Banach space without any generalized monotonicity by exploiting vector version of minimax inequality and obtain the existence results of solutions to the class of vector variational-like inequalities. The results presented here are different from [1, 5, 11], and extend and generalize the corresponding results in [7].

1. Introduction

A vector variational inequality in a finite-dimensional Euclidean space was first introduced by Giannessi [6] in 1980. This is a generalization of a scalar variational inequality to the vector case by virtue of multi-criterion consideration. Later on, vector variational inequalities have been investigated in abstract spaces, see [2, 3, 9]. It is

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worth noting that vector variational-like inequalities are important generalization of vector variational inequalities related to the class of η - connected sets which is much more general than the class of convex sets (see [8, 10, 11]). Moreover, Under the monotonicity conditions, the authors in [1, 5, 11] studied the vector variational (variational-like) inequalities by using K-Fan lemma. On the other, without any generalized monotonicity, the vector variational inequalities are studied by using the Brouwer and Browder fixed pointed theorems in [1, 5], respectively and in [7], Lai and Yao studied the existence of solutions of the vector variational inequalities by minimax inequality due to Fan [4].

Inspired and motivated by the above research work, we study the existence of the solutions of vector variational-like inequalities without any monotonicity by using vector version of minimax inequality. The results obtained in this paper are different from the corresponding results in [1, 5, 11] and extend and generalize the corresponding results in [7].

2. Preliminaries

Let X be a Banach space. A nonempty subset P of X is called a *pointed, convex cone* if $P + P \subset P$, $tP \subset P$ for all $t \geq 0$ and $P \cap (-P) = \{0\}$. The partial order "$\leq$" on X induced by a pointed cone is defined by declaring $x \leq y$ if and only if $y - x \in P$ for all $x, y \in X$, and in this case P is called a *positive cone* in X. Furthermore, if such a partial order is induced by a convex cone, it is called a *linear order*. A ordered Banach space is a pair (X, P), where X is a real Banach space and P is a pointed convex cone. With linear order induced by P, the weak order "$\not<$" on ordered Banach space (X, P) with $int\, P \neq \emptyset$ is defined as $x \not< y$ if and only if $y - x \notin int\, P$ for all $x, y \in X$ where "*int*" denotes the interior.

Let X and Y be real Banach spaces. $L(X, Y)$ is the space of all bounded linear mappings from X into Y. We denote by (l, x) the value of $l \in L(X, Y)$ at $x \in X$. Let K be a nonempty closed and convex subset of X, $T : K \to L(X, Y)$ be a single-valued mapping, and a set-valued mapping $C : K \to 2^Y$ be such that $C(x)$ is a closed, pointed and convex cone of Y with $intC(x) \neq \varnothing$ for all $x \in K$ and $\eta : K \times K \to X$ be two vector- valued mapping. In this paper, we consider the vector variational-like inequality problem, (denoted by VVIP) that is to find $x \in K$ such that

$$(Tx, \eta(y, x)) \notin -intC(x), \forall y \in K. \tag{2.1}$$

When $C(x) = P$ for all $x \in K$ and (Y, P) is an ordered Banach space with weak order, (VVLI) becomes $(\text{VVLI})'$, that is to find $x \in K$ such that

$$(Tx, \eta(y, x)) \nless 0, \quad \forall y \in K. \tag{2.2}$$

Furthermore, when $\eta(y, x) = y - x$, (VVLI) reduces to (VVI), that is to find $x \in K$ such that

$$(Tx, y - x) \notin -intC(x), \quad \forall\, y \in K. \tag{2.3}$$

Lai and Yao [7] stduied the existence of solution of vector variational-like inequalities (2.3) by minimax inequality in the case of nonmonotonicity conditions. In our paper, we study the existence results of (VVLI), which extend and generalize the results of [7] and different from the results of [1, 5, 11].

3. Main Results

In this section, we state and prove the existence results for vector variational-like inequalities without any generalized monotonicity assumption. To this end, the following result will be used.

Lemma 3.1 [4]**.** *Let E be a nonempty compact convex set of a Hausdorff topological vector space. Let A be a subset of $E \times E$ having the following properties*:

(i) $(x, x) \in A$ *for all* $x \in K$;

(ii) *for each* $x \in E$, *the set* $A_x = \{y \in E \mid (x, y) \in A\}$ *is closed in* E;

(iii) *for each* $y \in E$, *the set* $A_y = \{x \in E \mid (x, y) \notin A\}$ *is convex.*

Then there exists $y_0 \in E$ *such that* $E \times \{y_0\} \subset A$.

Now we can state and prove the main results of this paper.

Theorem 3.2. *Let X and Y be real Banach spaces. Let* K *be nonempty weakly compact convex subset of X. Let* $C : K \to 2^Y$ *be a set-valued mapping such that for all* $x \in K$, $C(x)$ *is a closed, pointed and convex cone in Y with* $\operatorname{int} C(x) \neq \emptyset$, *and a set-valued mapping* $W : K \to 2^Y$ *be defined by* $W(x) = Y \setminus (-intC(x))$ *such that the graph of W denoted by* $gphW$ *is weakly closed in* $X \times Y$. *Let* $T : K \to L(X, Y)$ *be a single-valued mapping such that for all* $x \in K$, *the mapping* $y \mapsto (Ty, \eta(x, y))$ *is continuous from the weak topology of K to the weak topology of Y. Let* $\eta : K \times K \to X$ *be a vector-valued mapping such that*

(a) $\eta(x, x) = 0, \quad \forall x \in K$;

(b) $\eta(x, y)$ *is affine with respect to x if, for any given* $y \in K$,

$$\eta(tx_1 + (1-t)x_2, y) = t\eta(x_1, y) + (1-t)\eta(x_2, y), \quad \forall x_1, x_2 \in K, t \in R,$$

with $x = tx_1 + (1-t)x_2 \in K$. *Then there exists* $x_0 \in K$ *such that*

$$(Tx_0, \eta(x, x_0)) \notin -intC(x_0), \quad \forall x \in K.$$

Proof. Let $A = \{(x, y) \in K \times K | (Ty, \eta(x, y)) \notin -intC(y)\}$. Then, it is clear that $(x, x) \in A$ for each $x \in K$. Next we show that for each $x \in K$, the set $A_x = \{y \in K | (x, y) \in A\}$ is weakly closed. To this end, let $\{y_\alpha\}$ be a net in A_x converging weakly to some $y \in K$. For each α, since $(x, y_\alpha) \in A$, we have

$$(Ty_\alpha, \eta(x, y_\alpha)) \notin -intC(y\alpha) \text{ or } (Ty_\alpha, \eta(x, y_\alpha)) \in y \setminus (-intC(y\alpha)).$$

By assumption, $(Ty_\alpha, \eta(x, y_\alpha))$ converges weakly to $(Ty, \eta(x, y))$. Since $gphW$ is weakly closed in $X \times Y$ we have

$$(Ty, \eta(x, y)) \in Y \setminus (-\operatorname{int} C(y)) \quad \text{or} \quad (Ty, \eta(x, y)) \notin -\operatorname{int} C(y).$$

Thus, $y \in A_x$ and consequently A_x is weakly closed.

Finally, we show that for each $y \in K$, the set $A_y = \{x \in K | (x, y) \notin A\}$ is convex. To this end, let $x_1, x_2 \in A_y$ and $t_1 \geq 0, t_2 \geq 0$ with $t_1 + t_2 = 1$. Then, A_y is convex. Since

$$(Ty, t_1\eta(x_1, y)) \in -int\, C(y),$$

$$(Ty, t_2\eta(x_2, y)) \in -int\, C(y).$$

As $C(y)$ is convex cone and the condition of (b), we have

$$(Ty, \eta(t_1x_1 + t_2x_2, y)) \in -int\, C(y)$$

hence, $t_1x_1 + t_2x_2 \in A_y$, and therefore A_y is convex.

Now by invoking Lemma 3.1, there exists $x_0 \in K$ such that $K \times \{x_0\} \subset A$. This implies that $x_0 \in K$ and

$$(Tx_0, \eta(x, x_0)) \notin -intC(x_0) \quad \forall x \in K,$$

which implies that the (VVLI) has a solution. This completes the proof.

We can derive the following corollary from Theorem 3.2.

Corollary 3.3. *Let X and Y be real Banach spaces. Let K be a nonempty compact convex subset of X. Let* $C : K \to 2^Y$ *be a set-valued mapping such that for each* $x \in K$, $C(x)$ *is a closed pointed and convex cone and* $int\, C(x) \neq \varnothing$, *and* $W : K \to 2^Y$ *be defined by* $W(x) = Y \setminus (- int\, C(x)$ *such that* $gphW$ *is weakly closed in* $X \times Y$. *Let* $T : K \to L(X, Y)$ *be continuous from the weak topology of K to the norm topology of Y. Let* $\eta : K \times K \to X$ *be such that*

(a) $\eta(x, x) = 0, \quad \forall x \in K$;

(b) $\eta(x, y)$ *is affine with respect to x if, for any given* $y \in K$,

$$\eta(tx_1 + (1-t)x_2, y) = t\eta(x_1, y) + (1-t)\eta(x_2, y), \forall x_1, x_2 \in K, t \in R,$$

with $x = tx_1 + (1-t)x_2 \in K$. *Then there exists* $x_0 \in K$ *such that*

$$(Tx_0, \eta(x, x_0)) \notin -int\, C(x_0) \quad \textit{for all } x \in K;$$

(c) $\forall x \in K$, $\eta(x, y)$ *is weakly continuous in the first argument*.

Then there exists $x_0 \in K$ *such that*

$$(Tx_0, \eta(x, x_0)) \notin -intC(x_0), \forall x \in K.$$

Proof. It suffices to check that for each $x \in K$, the mapping $y \mapsto (Ty, \eta(x, y))$ is continuous from weak topology of K to the weak topology of Y. To this end, let $x \in K$ be arbitrary but fixed, and let $T_x : K \to Y$ be defined by $T_x y = (Ty, \eta(x, y))$, $\forall y \in K$. Let $\{y_\alpha\}$ be any net in K converging weakly to some $y \in K$. By assumption, we have

$\|Ty_\alpha - Ty\|_{L(X,Y)} \to 0$. Since the net $\{y_\alpha\}$ is weakly convergent and the condition of (c), it is bounded. Therefore,

$$|(Ty_\alpha - Ty, \eta(x, y\alpha))| \leq \| Ty_\alpha - Ty\|_{L(X, Y)} \| \eta(x, y_\alpha)\|_X \to 0$$

and hence $(Ty_\alpha - Ty, \eta(x, y_\alpha))$ converges weakly to 0 in Y. On the other hand, as $Ty \in L(X, Y)$, Ty is continuous from the weak topology of X to the weak topology of Y. Consequently, we have

$$T_x y_\alpha = (Ty_\alpha, \eta(x, y_\alpha)) = (Ty_\alpha - Ty, \eta(x, y_\alpha)) + (Ty, \eta(x, y_\alpha))$$

converges weakly to $(Ty, \eta(x, y)) = T_x y$. Hence the operator T_x is continuous from the weak topology of K to the weak topology of Y. The result then follows from Theorem 3.2.

From Corollary 3.3, we have the following result.

Corollary 3.4. *Let X be a real Banach space*, (Y, C) *be an ordered Banach space*, *where C is a pointed*, *closed and convex cone in Y with* $int\, C(x) \neq \emptyset$, *such that* $Y \setminus (-intC)$ *is weakly closed*. *Let* K, T, η *be as in Corollary* 3.3. *Then there exists* $x_0 \in K$ *such that*

$$(Tx_0, \eta(x, x_0)) \nless 0, \forall x \in K.$$

Remark 3.5. Theorem 3.2, Corollaries 3.3 and 3.4 extend and generalize the corresponding results in [7].

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